

A STRUCTURE-PRESERVING AND THERMODYNAMICALLY COMPATIBLE CELL-CENTERED LAGRANGIAN FINITE VOLUME SCHEME FOR CONTINUUM MECHANICS*

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Abstract. In this work we present a novel structure-preserving scheme for the compatible discretization of the Godunov–Peshkov–Romenski (GPR) model of continuum mechanics written in Lagrangian form. The governing equations fall into the larger class of overdetermined hyperbolic and thermodynamically compatible systems of partial differential equations (PDE). This model admits an extra conservation law for the total energy (first principle of thermodynamics) and satisfies the entropy inequality (second principle of thermodynamics). Furthermore, in the absence of algebraic source terms, the distortion field of the continuum and the specific thermal impulse satisfy a curl-free condition, provided the initial data are curl-free. Last but not least, the determinant of the distortion field is related to the density of the medium, i.e., the system is also endowed with a nonlinear algebraic constraint. In the stiff relaxation limit, the system tends to the compressible Navier–Stokes equations, i.e., the GPR model is able to describe at the same time the dynamics of nonlinear solids as well as the one of fluids. The objective of this work is to construct and analyze a new semidiscrete thermodynamically compatible cell-centered Lagrangian finite volume scheme on moving unstructured meshes that satisfies the following structural properties of the governing PDE exactly at the discrete level: (i) compatibility with the first law of thermodynamics, i.e., discrete total energy conservation; (ii) compatibility with the second law of thermodynamics, i.e., discrete entropy inequality; (iii) exact discrete compatibility between the density and the determinant of the distortion field; (iv) exact preservation of the curl-free property of the distortion field and of the specific thermal impulse in the absence of algebraic source terms. We will show that it is indeed possible to achieve all the above properties simultaneously. Unlike in existing schemes, we choose to directly discretize the entropy inequality, hence obtaining total energy conservation as a consequence of an appropriate and thermodynamically compatible discretization of all the other equations. From this discretization, property (ii) above follows trivially by construction, while (i) leads to provable nonlinear stability, which is an important feature, in particular for the complex PDE system under consideration here. The thermodynamic compatibility and thus nonlinear stability in the sense of total energy conservation is achieved via a very simple and general approach recently introduced by Abgrall, Dumbser, and Maire. by using a scalar correction factor that is defined at the nodes of the grid. This perfectly fits into the formalism of nodal solvers which is typically adopted in cell-centered

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Lagrangian finite volume methods. The new scheme is run on some academic benchmark problems for computational fluid and solid mechanics to show that the properties also hold in the practical implementation of the scheme.

Key words. thermodynamically compatible finite volume schemes, Lagrangian continuum mechanics, cell entropy inequality, nonlinear stability, exact preservation of determinant and curl constraints, moving unstructured mesh

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1. Introduction. In [27] Godunov discovered for the first time the connection between symmetric hyperbolicity in the sense of Friedrichs [25] and thermodynamic compatibility with the first and second laws of thermodynamics. 10 years later, the same connection was rediscovered by Friedrichs and Lax in [26]. Subsequently, Godunov and collaborators extended this discovery to the more general formalism of symmetric hyperbolic and thermodynamically compatible (SHTC) systems, including compressible magnetohydrodynamics, nonlinear hyperelasticity, and compressible multiphase flows; see, e.g., [28, 30, 29, 42, 43]. In his seminal papers [47, 48] Tadmor was the first to achieve thermodynamic compatibility also on the discrete level inside a numerical scheme, obtaining the discrete compatibility with the *entropy inequality* (second principle of thermodynamics) as a consequence of the compatible numerical scheme, while directly discretizing the *total energy* conservation law (first principle of thermodynamics). Further ongoing research on this subject can be found, for example, in [48, 31, 41, 23, 40, 18, 19, 34, 20]. Entropy-compatible discretizations for nonconservative hyperbolic equations were forwarded, for instance, in [22, 2, 1]. For hydrodynamics and continuum mechanics a first attempt to achieve discrete total energy conservation as a consequence of all other equations rather than a discrete entropy inequality was made in [15]. In the context of updated Lagrangian schemes for hydrodynamics on moving meshes, total energy conservation can be obtained as a consequence of the discrete mass, momentum, and internal energy equations; see, e.g., [16, 6]. Further developments on total energy conserving and entropy-compatible Lagrangian schemes for hydrodynamics can be found in [17, 35, 14, 37, 11].

From a mathematical point of view, a particularly interesting system is the system of nonlinear hyperelasticity, which was written in Eulerian coordinates in [30]. It was later extended to incorporate also strain relaxation source terms that allow us to describe viscoplastic solids and viscous fluids in [39, 21]. Compared to standard hydrodynamics, it includes two additional quantities that need to be evolved in time, namely, the distortion tensor field $\mathbf{A}$, which in the absence of source terms is the Eulerian gradient of the Lagrangian reference coordinates, and the specific thermal impulse $\mathbf{J}$, which allows for hyperbolic heat conduction. The PDE system is symmetrizable [42], but the energy potential is not strictly convex, hence the Hessian of the energy potential is obviously symmetric, but not strictly positive definite everywhere. However, the system has the following interesting additional mathematical properties:

1. the system is thermodynamically compatible, i.e., it satisfies total energy conservation and
2. it satisfies an entropy inequality;
3. the system admits an algebraic constraint that connects the determinant of the distortion field $\mathbf{A}$ with the mass density ρ of the medium;

4. in the absence of source terms, the distortion field $\mathbf{A}$ and the thermal impulse $\mathbf{J}$ satisfy the stationary differential constraints (involutions) $\nabla \times \mathbf{A} = 0$ and $\nabla \times \mathbf{J} = 0$.

The main objective of this paper is to construct a numerical scheme that satisfies simultaneously all the properties 1–4 exactly also on the discrete level. To the best of our knowledge, up to now there exists no such scheme. Previous numerical methods were either only compatible with the involutions [10], or only thermodynamically compatible [15, 3], or thermodynamically compatible and compatible with the determinant constraint [12], but none was able to satisfy all the properties 1–4 simultaneously. A very special feature of our scheme is that we discretize the entropy inequality directly and obtain total energy conservation as a consequence, similar to what was done in [15, 3, 11]. The rest of this paper is organized as follows. In section 2 we introduce the governing partial differential equations. In section 3 we present our new hyperbolic and thermodynamically compatible (HTC) Lagrangian scheme and prove that all the mathematical properties 1–4 also hold exactly on the semidiscrete level. Numerical results for some elementary test cases are shown in section 4. The paper closes with some conclusions and an outlook to future research in section 5.

2. A unified first order thermodynamically compatible model of continuum mechanics. Using the material derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$, the unified first order hyperbolic model of continuum mechanics of Godunov, Peshkov, and Romenski (GPR) [30, 42, 39, 21, 15], that is able to describe the dynamics of nonlinear elastic solids at large deformations and viscous fluids with heat conduction, can be written in the updated Lagrangian form [13, 8] as follows:

$$\begin{aligned}
 (2.1a) \quad & \rho \frac{d\tau}{dt} - \frac{\partial v_k}{\partial x_k} = 0, \\
 (2.1b) \quad & \rho \frac{dv_i}{dt} + \frac{\partial}{\partial x_k} (p\delta_{ik} + \sigma_{ik}) = 0, \\
 (2.1c) \quad & \rho \frac{dS}{dt} + \frac{\partial(\rho\beta_k)}{\partial x_k} = \frac{\Gamma_{ij}\Gamma_{ij}}{T\theta_1(\tau_1)} + \frac{\beta_i\beta_i}{T\theta_2(\tau_2)} \geq 0, \\
 (2.1d) \quad & \rho \frac{dA_{ik}}{dt} + \rho A_{im} \frac{\partial v_m}{\partial x_k} = -\frac{1}{\theta_1(\tau_1)} \Gamma_{ik}, \\
 (2.1e) \quad & \rho \frac{dJ_i}{dt} + \rho J_m \frac{\partial v_m}{\partial x_i} + \rho \frac{\partial T}{\partial x_i} = -\frac{1}{\theta_2(\tau_2)} \beta_i.
 \end{aligned}$$

Here and in the following, we employ the usual Einstein summation convention, which implies summation over repeated indices. Furthermore, we consider the nondimensional model whose derivation is shown in Appendix A. The above model is supplemented by the following extra conservation law for the total energy (see [42, 39, 21, 15]):

$$(2.2) \quad \rho \frac{dE}{dt} + \frac{\partial}{\partial x_k} (v_i (p\delta_{ik} + \sigma_{ik}) + \rho T \beta_k) = 0.$$

Here, $\mathbf{x} \in \mathbb{R}^d$ is the spatial position vector in d space dimensions, $t \in \mathbb{R}^+$ is the time coordinate, and $\mathbf{A} = \{A_{ik}\}$ with $A_{ik} = \partial \xi_i / \partial x_k$ is the Eulerian gradient of the Lagrangian position variable ξ_i , representing the reference configuration. Furthermore, ρ is the fluid density and $\tau = 1/\rho$ is the specific volume, S is the specific entropy, $\mathbf{v} = \{v_i\} \in \mathbb{R}^d$ is the velocity vector, p denotes the fluid pressure, $\boldsymbol{\sigma} = \{\sigma_{ik}\}$ is the stress tensor, $\mathbf{J} = \{J_i\}$ is the specific thermal impulse, $\boldsymbol{\Gamma} = \{\Gamma_{ik}\}$ and $\boldsymbol{\beta} = \{\beta_k\}$ are thermodynamic dual variables that are defined later and $E = E_1 + E_2 + E_3 + E_4$ is the

specific total energy, which is a sum of the following contributions: the specific internal energy $E_1 = E_1(\tau, S)$; the kinetic energy $E_2 = \frac{1}{2}v_m v_m$; the energy contained in the elastic deformations $E_3 = \frac{1}{4}c_s^2 \mathring{G}_{ij} \mathring{G}_{ij}$ with the metric tensor $G_{ij} = A_{ki} A_{kj}$, its trace-free part $\mathring{G}_{ij} = G_{ij} - \frac{1}{3}G_{mm}\delta_{ij}$, and δ_{ij} the usual Kronecker symbol; $E_4 = \frac{1}{2}c_h^2 J_m J_m$ is the energy contained in the specific thermal impulse J_m , which is needed within a hyperbolic model of heat conduction. The constant c_s represents the shear sound speed in the medium and c_h is related to the finite propagation speed of heat waves. The functions $\theta_1 > 0$ and $\theta_2 > 0$ depend on two characteristic relaxation times $\tau_1 > 0$ and $\tau_2 > 0$ and are given by (see [21, 10, 15]),

$$(2.3) \quad \theta_1 = \frac{1}{3} \frac{\tau_1}{\rho} c_s^2 |\mathbf{A}|^{-\frac{5}{3}}, \quad \theta_2 = c_h^2 \frac{\tau_2}{\rho} \frac{T_0}{T},$$

where the reference values of density and temperature are those considered at the initial time $t = 0$, i.e., $\rho_0 = \rho(0)$ and $T_0 = T(0)$. From the above definitions it follows that $\theta_1 \rightarrow \infty$ as $\tau_1 \rightarrow \infty$ and $\theta_2 \rightarrow \infty$ as $\tau_2 \rightarrow \infty$. In the stiff relaxation limit, i.e., for $\tau_1 \rightarrow 0$ and $\tau_2 \rightarrow 0$, under the classical hypothesis of finite viscosity and heat conduction coefficient, the model recovers the classical Navier–Stokes stress tensor and the Fourier law for the heat flux; see [21, 9]. We recall that Lagrangian mass conservation is written

$$(2.4) \quad \frac{\rho}{|\mathbf{A}|} = \rho_0.$$

The equation of the specific volume (2.1a) is redundant since it might be obtained as the dot product of the equation of the distortion tensor (2.1d) with the comatrix $|\mathbf{A}|\mathbf{A}^{-1}$ and using the above mass conservation (2.4) [12]. Furthermore, in the updated Lagrangian framework, the system is supplemented with the trajectory equation for the Eulerian coordinate as

$$(2.5) \quad \frac{d\mathbf{x}}{dt} = \mathbf{v}, \quad \mathbf{x}(0) = \boldsymbol{\xi},$$

where the Lagrangian coordinate $\boldsymbol{\xi}$ of the initial reference configuration corresponds to the Eulerian coordinate $\mathbf{x}$ at $t = 0$. The system is closed by an equation of state (EOS) for the specific internal energy E_1 in terms of the specific volume τ and the specific entropy S , hence $E_1 = E_1(\tau, S)$. In the following we assume that $E_1(\tau, S)$ is convex in order to assure thermodynamic stability of the system; see [38]. Following [27, 42, 21, 15, 11] the thermodynamic *dual variables*, or *Godunov variables* [24], i.e., the *main field* according to [44] reads as follows:

$$(2.6) \quad p = -\frac{\partial E}{\partial \tau}, \quad \mathbf{v} = \frac{\partial E}{\partial \mathbf{v}}, \quad T = \frac{\partial E}{\partial S}, \quad \boldsymbol{\Gamma} = \frac{\partial E}{\partial \mathbf{A}}, \quad \boldsymbol{\beta} = \frac{\partial E}{\partial \mathbf{J}}.$$

For convenience we also introduce the vector of dual variables $\mathbf{w} = (-p, \mathbf{v}, T, \boldsymbol{\Gamma}, \boldsymbol{\beta})$. Based on these definitions, the stress tensor can be expressed as follows:

$$(2.7) \quad \boldsymbol{\sigma} = \rho \mathbf{A}^T \boldsymbol{\Gamma} + \rho \mathbf{J} \otimes \boldsymbol{\beta} \quad \text{or, in index notation,} \quad \sigma_{ik} = \rho A_{ji} \Gamma_{jk} + \rho J_i \beta_k.$$

We furthermore assume that the temperature is positive, $T > 0$. After some calculations one can check that the total energy conservation law (2.2) can be obtained as a consequence of (2.1) after multiplication of the governing PDE system with the main field variables $\mathbf{w}$ and subsequent summation. In the next section, we design a numerical method that can preserve this thermodynamic compatibility also on the discrete level.

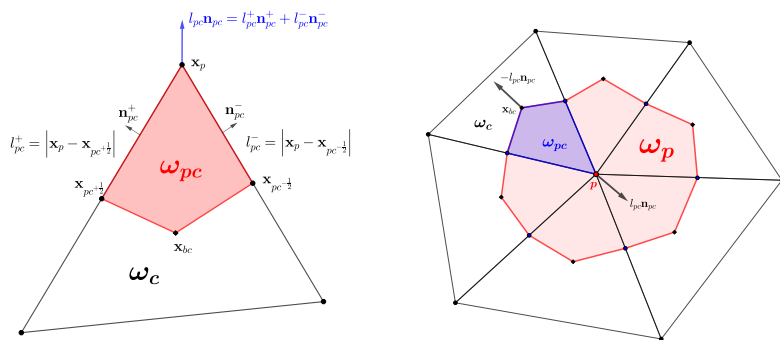


FIG. 1. Notation for the cell ω_c (left) and definition of the dual cell ω_p (right). Note: color appears only in the online article.

3. Numerical method. The computational domain $\Omega(t)$ with boundary $\partial\Omega(t)$ in $d = 2$ space dimensions is discretized with a set of nonoverlapping triangular control volumes $\omega_c(t)$ of volume $|\omega_c(t)|$. The quantities related to the mesh geometry are time dependent since the mesh moves, but the topology of the grid remains unchanged. A generic vertex of the mesh is denoted by p , while the subcell ω_{pc} refers to the portion of cell ω_c attached to one of its vertices p , as depicted in Figure 1. For the sake of clarity, the vertex index will be always written in subscript notation, which is meant not to be misunderstood with the pressure variable. The set of vertices belonging to cell ω_c is referred to with $\mathcal{P}(c)$, while the set of cells sharing node p is indicated with $\mathcal{C}(p)$. The subcell ω_{pc} is defined by connecting the vertex with position $\mathbf{x}_p$, the cell barycenter of coordinates $\mathbf{x}_{bc} = 1/(d+1) \sum_{p \in \mathcal{P}(c)} \mathbf{x}_p$, and the left and right midpoints of the edges impinging on node p , namely, $pc-\frac{1}{2}$ and $pc+\frac{1}{2}$, respectively. The half-lengths of these edges are given by $l_{pc}^- = |\mathbf{x}_p - \mathbf{x}_{pc-\frac{1}{2}}|$ and $l_{pc}^+ = |\mathbf{x}_p - \mathbf{x}_{pc+\frac{1}{2}}|$, and the corner normal is computed as

$$(3.1) \quad l_{pc}\mathbf{n}_{pc} = l_{pc}^+\mathbf{n}_{pc}^+ + l_{pc}^-\mathbf{n}_{pc}^-.$$

Thanks to Gauss' theorem the corner normals satisfy the identity

$$(3.2) \quad \sum_{p \in \mathcal{P}(c)} l_{pc}\mathbf{n}_{pc} = \mathbf{0},$$

and they can be immediately used to discretize the differential operators appearing in the governing model (2.1). In Figure 1 we also introduce the dual cell ω_p given by the union of all subcells sharing a generic node p . The outward pointing corner vectors of ω_p are simply the corner normals $l_{pc}\mathbf{n}_{pc}$ with opposite sign, hence also the following identity holds:

$$(3.3) \quad \sum_{c \in \mathcal{C}(p)} -l_{pc}\mathbf{n}_{pc} = \mathbf{0}.$$

3.1. Semidiscrete thermodynamically compatible Lagrangian scheme.

In the following we propose a cell-centered finite volume scheme that is by construction exactly compatible with the entropy inequality (2.1c) and which satisfies the extra conservation law of total energy conservation (2.2). Let $m_c = |\omega_c|\rho_c = |\omega_c|\tau_c^{-1}$ be the mass of the cell, which remains constant in the Lagrangian framework [36], and let the mass-averaged cell value of a generic quantity $q(\mathbf{x}, t)$ be defined as

$$(3.4) \quad q_c = \frac{1}{m_c} \int_{\omega_c(t)} \rho q \, d\mathbf{x}.$$

Integration of system (2.1) over the cell $\omega_c(t)$, using the Reynolds transport formula and suitable numerical fluctuations in combination with (3.2), yields the following first order in space semidiscrete Lagrangian HTC scheme:

$$(3.5a) \quad m_c \frac{d\tau_c}{dt} - \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \cdot (\mathbf{v}_p - \mathbf{v}_c) = 0,$$

$$(3.5b) \quad m_c \frac{d\mathbf{v}_c}{dt} + \sum_{p \in \mathcal{P}(c)} (l_{pc} \mathbf{n}_{pc} (p_p - p_c) + l_{pc} \mathbf{n}_{pc} \cdot (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_c)) \\ + \sum_{p \in \mathcal{P}(c)} l_{pc} (\alpha_p + \epsilon_p) (\mathbf{v}_c - \mathbf{v}_p) = \mathbf{0},$$

$$(3.5c) \quad m_c \frac{dS_c}{dt} + \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \cdot (\rho_p \boldsymbol{\beta}_p - \rho_c \boldsymbol{\beta}_c) = \Pi_c + \pi_c \geq 0.$$

$$(3.5d) \quad m_c \frac{d\mathbf{A}_c}{dt} + \rho_c \mathbf{A}_c \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} (\mathbf{v}_p - \mathbf{v}_c) = -\frac{1}{\theta_1(\tau_1)} \boldsymbol{\Gamma}_c,$$

$$(3.5e) \quad m_c \frac{d\mathbf{J}_c}{dt} + \rho_c \mathbf{J}_c \cdot \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} (\mathbf{v}_p - \mathbf{v}_c) + \rho_c \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} (T_p - T_c) \\ + \rho_c \sum_{p \in \mathcal{P}(c)} (l_{pc} \mathbf{n}_{pc} (\alpha_p + \epsilon_p) |\omega_p| \nabla_p^c \cdot (\rho_c \boldsymbol{\beta}_c)) = -\frac{1}{\theta_2(\tau_2)} \boldsymbol{\beta}_c$$

with $\sum_p l_{pc} \mathbf{n}_{pc} \cdot \boldsymbol{\sigma}_p = \sum_p l_{pc} n_{pc}^k \cdot \sigma_p^{ik}$ and $\mathbf{A}_c \sum_p l_{pc} \mathbf{n}_{pc} \tilde{\mathbf{v}}_p = A_c^{im} \sum_p l_{pc} n_{pc}^k \tilde{v}_p^m$, where n_{pc}^k is the k th component of the unit normal vector $\mathbf{n}_{pc}$. As in [11] we have followed the general framework of entropy conservative schemes proposed by Abgrall, Busto, and Dumbser. in [1, 3], with the scalar correction factor $\alpha_p \in \mathbb{R}$ that ensures discrete thermodynamic compatibility with the total energy conservation law and which will be defined later. $\epsilon_p \geq 0$ is a numerical viscosity that can be switched off to obtain an entropy conserving Lagrangian scheme. The object $|\omega_p| \nabla_p^c \cdot \rho_c \boldsymbol{\beta}_c$ is a volume-integrated discrete divergence defined for each node p as

$$(3.6) \quad |\omega_p| \nabla_p^c \cdot \rho_c \boldsymbol{\beta}_c = - \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot \rho_c \boldsymbol{\beta}_c.$$

Furthermore, ρ_p , p_p , T_p are the averaged density, pressure, and temperature in the node p , while $\boldsymbol{\sigma}_p$ and $\boldsymbol{\beta}_p$ are node-averages of the stress tensor and of the thermodynamic dual variable to the thermal impulse, respectively. The mesh motion is governed by the trajectory equation (2.5) that is semidiscretized at each node of the mesh as

$$(3.7) \quad \frac{d\mathbf{x}_p}{dt} = \mathbf{v}_p, \quad \mathbf{x}_p(0) = \boldsymbol{\xi}_p(0).$$

The nodal values of ρ_p , $\mathbf{v}_p$, p_p , T_p , $\boldsymbol{\beta}_p$, and $\boldsymbol{\sigma}_p$ are obtained via conservation principles, i.e., requiring that the sum of the fluctuations around a node is equal to the sum of the fluxes through the boundary of the dual cell ω_p , hence, to achieve conservation in the discrete momentum equation (3.5b) we require

$$\begin{aligned}
 (3.8) \quad & \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} (p_p - p_c) + \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \cdot (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_c) + \alpha_p \sum_{c \in \mathcal{C}(p)} l_{pc} (\mathbf{v}_c - \mathbf{v}_p) \\
 &= - \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} p_c - \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot \boldsymbol{\sigma}_c = \int_{\partial \omega_p(t)} (p \mathbf{I} + \boldsymbol{\sigma}) \cdot \mathbf{n} \, ds.
 \end{aligned}$$

Using (3.3), we obtain the nodal velocity that satisfies conservation:

$$(3.9) \quad \mathbf{v}_p = \frac{\sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{v}_c}{\sum_{c \in \mathcal{C}(p)} l_{pc}}.$$

For ρ_p , p_p , T_p , $\boldsymbol{\beta}_p$, and $\boldsymbol{\sigma}_p$ the choice of the corner values is arbitrary, since conservation is automatically guaranteed, so we simply choose the same average as above, i.e.,

$$(3.10) \quad p_p = \frac{\sum_{c \in \mathcal{C}(p)} l_{pc} p_c}{\sum_{c \in \mathcal{C}(p)} l_{pc}}, \quad \rho_p = \frac{\sum_{c \in \mathcal{C}(p)} l_{pc} \rho_c}{\sum_{c \in \mathcal{C}(p)} l_{pc}}, \quad T_p = \frac{\sum_{c \in \mathcal{C}(p)} l_{pc} T_c}{\sum_{c \in \mathcal{C}(p)} l_{pc}},$$

$$(3.11) \quad \boldsymbol{\sigma}_p = \frac{\sum_{c \in \mathcal{C}(p)} l_{pc} \boldsymbol{\sigma}_c}{\sum_{c \in \mathcal{C}(p)} l_{pc}}, \quad \boldsymbol{\beta}_p = \frac{\sum_{c \in \mathcal{C}(p)} l_{pc} \boldsymbol{\beta}_c}{\sum_{c \in \mathcal{C}(p)} l_{pc}}.$$

As is done in [11], we now require thermodynamic compatibility by imposing the following condition around a node p : the sum of the total energy fluctuations around a node must be equal to the sum of total energy fluxes around the same node. The total energy fluctuations are obtained as the dot product of the fluctuations in the scheme (3.5) with the main-field variables $\mathbf{w}_c$. This leads to the following nodal condition:

$$\begin{aligned}
 (3.12) \quad & \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (p_c (\mathbf{v}_p - \mathbf{v}_c) + \mathbf{v}_c (p_p - p_c) + (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_c) \mathbf{v}_c + T_c (\rho_p \boldsymbol{\beta}_p - \rho_c \boldsymbol{\beta}_c)) \\
 &+ \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (\rho_c \boldsymbol{\Gamma}_c \mathbf{A}_c (\mathbf{v}_p - \mathbf{v}_c) + \rho_c \boldsymbol{\beta}_c \otimes \mathbf{J}_c (\mathbf{v}_p - \mathbf{v}_c) + \rho_c \boldsymbol{\beta}_c (T_p - T_c)) \\
 &+ \alpha_p \left(\sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{v}_c \cdot (\mathbf{v}_c - \mathbf{v}_p) + \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot \rho_c \boldsymbol{\beta}_c (|\omega_p| \nabla_p^c \cdot \rho_c \boldsymbol{\beta}_c) \right) \\
 &= - \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (p_c \mathbf{v}_c + \boldsymbol{\sigma}_c \mathbf{v}_c + \rho_c T_c \boldsymbol{\beta}_c) = \int_{\partial \omega_p(t)} (p \mathbf{v} + \boldsymbol{\sigma} \mathbf{v} + \rho T \boldsymbol{\beta}) \cdot \mathbf{n} \, ds.
 \end{aligned}$$

The terms multiplied by the scalar factor α_p can be conveniently rearranged in quadratic form using the definitions (3.9) and (3.6) (see also [11]):

$$\begin{aligned}
 (3.13) \quad & \delta_p := \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{v}_c \cdot (\mathbf{v}_c - \mathbf{v}_p) + \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot \rho_c \boldsymbol{\beta}_c (|\omega_p| \nabla_p^c \cdot \rho_c \boldsymbol{\beta}_c) \\
 &= \left(\sum_{c \in \mathcal{C}(p)} l_{pc} (\mathbf{v}_c - \mathbf{v}_p)^2 \right) + (|\omega_p| \nabla_p^c \cdot \rho_c \boldsymbol{\beta}_c)^2 \geq 0.
 \end{aligned}$$

Solving (3.12) for α_p one obtains

$$(3.14) \quad \alpha_p = \frac{\nu_p}{\delta_p}$$

with the denominator given in (3.13) and the numerator by

$$(3.15) \quad \begin{aligned} \nu_p = & \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (p_c \mathbf{v}_c - p_c \mathbf{v}_p - p_p \mathbf{v}_c) + \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (\boldsymbol{\sigma}_c \mathbf{v}_c - \boldsymbol{\sigma}_c \mathbf{v}_p - \boldsymbol{\sigma}_p \mathbf{v}_c) \\ & + \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (T_c \rho_c \boldsymbol{\beta}_c - T_c \rho_p \boldsymbol{\beta}_p - T_p \rho_c \boldsymbol{\beta}_c). \end{aligned}$$

For $\delta_p = 0$ we set $\alpha_p = 0$. The two entropy production terms read as follows: the one related to the algebraic source term is simply given by

$$(3.16) \quad \pi_c = \frac{\boldsymbol{\Gamma}_c : \boldsymbol{\Gamma}_c}{T_c \theta_1(\tau_1)} + \frac{\boldsymbol{\beta}_c \cdot \boldsymbol{\beta}_c}{T_c \theta_2(\tau_2)} \geq 0,$$

which is quadratic in $\boldsymbol{\Gamma}_c$ and $\boldsymbol{\beta}_c$ and clearly nonnegative since $T_c > 0$ and $(\theta_1, \theta_2) > 0$ is assumed. The entropy production term related to the numerical viscosity $\epsilon_p \geq 0$, which may be introduced into the numerical scheme, reads

$$(3.17) \quad \Pi_c = \sum_{p \in \mathcal{P}(c)} \frac{|\omega_{pc}|}{|\omega_c|} \frac{\Pi_p}{T_c}$$

with the nodal numerical entropy production

$$(3.18) \quad \Pi_p = \epsilon_p \left(\sum_{c \in \mathcal{C}(p)} l_{pc} (\mathbf{v}_c - \mathbf{v}_p)^2 \right) + \epsilon_p (|\omega_p| \nabla_p^c \cdot \rho_c \boldsymbol{\beta}_c)^2 = \epsilon_p \delta_p \geq 0,$$

which is also obviously nonnegative. Since by assumption $T_c > 0$, also $\Pi_c \geq 0$. Hence, in our scheme, we immediately have a discrete cell entropy inequality, *by construction*.

As we prove now, the semidiscrete scheme (3.5) is also nonlinearly stable in the sense that it conserves total energy exactly, it satisfies the determinant constraint $|\mathbf{A}_c| = \rho_c / \rho_c(0)$ at a semidiscrete level, and preserves the curl-free property of $\mathbf{J}_c$ and $\mathbf{A}_c$ for initially curl-free data and vanishing sources ($\tau_1 \rightarrow \infty$ and $\tau_2 \rightarrow \infty$). To show the involution-preserving properties we first prove the following lemma; see also [5, 46, 4].

LEMMA 3.1. *The discrete gradient operator applied to a scalar field ϕ_p defined in the nodes p ,*

$$(3.19) \quad \nabla_c^p \phi_p = \frac{1}{|\omega_c|} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \phi_p,$$

and its dual discrete curl operator applied to a vector field $\mathbf{J}_c$ defined in the cell centers,

$$(3.20) \quad \nabla_p^c \times \mathbf{J}_c = -\frac{1}{|\omega_p|} \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \times \mathbf{J}_c,$$

satisfy the discrete vector calculus identity

$$(3.21) \quad \nabla_p^c \times \nabla_c^p \phi_p = 0.$$

Proof. In the following we denote the indices of the three nodes of control volume ω_c by p_1 , p_2 , and p_3 , with Eulerian node coordinates $\mathbf{x}_{p_1}$, $\mathbf{x}_{p_2}$, and $\mathbf{x}_{p_3}$, respectively. We furthermore assume that the nodes are ordered counterclockwise. The indices of

the edges of ω_c are denoted by e_1 , e_2 , and e_3 , respectively, with edge e_1 composed of nodes p_1 and p_2 , edge e_2 composed of nodes p_2 and p_3 , and edge e_3 composed of nodes p_3 and p_1 , respectively. Assuming a linear distribution of the scalar field ϕ_p inside control volume ω_c (according to continuous P1 Lagrange finite elements) immediately yields the following averages in the edge midpoints:

$$(3.22) \quad \phi_{e_1} = \frac{1}{2}(\phi_{p_1} + \phi_{p_2}), \quad \phi_{e_2} = \frac{1}{2}(\phi_{p_2} + \phi_{p_3}), \quad \phi_{e_3} = \frac{1}{2}(\phi_{p_3} + \phi_{p_1}).$$

It can easily be checked that the three corner normals are given by

$$(3.23) \quad \begin{aligned} l_{p_1 c} \mathbf{n}_{p_1 c} &= \frac{1}{2}((y_{p_2} - y_{p_3}), (x_{p_3} - x_{p_2}), 0)^T, \\ l_{p_2 c} \mathbf{n}_{p_2 c} &= \frac{1}{2}((y_{p_3} - y_{p_1}), (x_{p_1} - x_{p_3}), 0)^T, \\ l_{p_3 c} \mathbf{n}_{p_3 c} &= \frac{1}{2}((y_{p_1} - y_{p_2}), (x_{p_2} - x_{p_1}), 0)^T, \end{aligned}$$

while the area of element ω_c is

$$(3.24) \quad |\omega_c| = \frac{1}{2}(x_{p_1}y_{p_2} - x_{p_1}y_{p_3} - x_{p_2}y_{p_1} + x_{p_2}y_{p_3} + x_{p_3}y_{p_1} - x_{p_3}y_{p_2}).$$

The discrete gradient $\nabla_c^p \phi_p = \frac{1}{|\omega_c|} \sum_p l_{pc} \mathbf{n}_{pc} \phi_p$ of the scalar field ϕ_p in the cell c then reads

$$(3.25) \quad \nabla_c^p \phi_p = \frac{1}{2|\omega_c|} \begin{pmatrix} (y_{p_2} - y_{p_3})\phi_{p_1} + (y_{p_3} - y_{p_1})\phi_{p_2} + (y_{p_1} - y_{p_2})\phi_{p_3} \\ (x_{p_3} - x_{p_2})\phi_{p_1} + (x_{p_1} - x_{p_3})\phi_{p_2} + (x_{p_2} - x_{p_1})\phi_{p_3} \\ 0 \end{pmatrix}.$$

Since this property will be needed later, we now also show that this gradient is identical to the discrete gradient obtained by using continuous P1 Lagrange finite elements inside the control volume ω_c . The associated basis functions in the universal reference element $T_0 = \{\zeta_1, \zeta_2 : 0 \leq \zeta_1 \leq 1, 0 \leq \zeta_2 \leq 1 - \zeta_1\}$ with nodes $(0, 0)$, $(1, 0)$, and $(0, 1)$ and reference coordinates $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ read

$$(3.26) \quad \psi_{p_1} = \psi_{p_1}(\boldsymbol{\zeta}) = 1 - \zeta_1 - \zeta_2, \quad \psi_{p_2} = \psi_{p_2}(\boldsymbol{\zeta}) = \zeta_1, \quad \psi_{p_3} = \psi_{p_3}(\boldsymbol{\zeta}) = \zeta_2.$$

The Jacobian of the mapping

$$(3.27) \quad \mathbf{x} = \mathbf{x}(\boldsymbol{\zeta}) = \psi_{p_1}(\boldsymbol{\zeta}) \mathbf{x}_{p_1} + \psi_{p_2}(\boldsymbol{\zeta}) \mathbf{x}_{p_2} + \psi_{p_3}(\boldsymbol{\zeta}) \mathbf{x}_{p_3} = \sum_{p \in \mathcal{P}(c)} \psi_p \mathbf{x}_p$$

reads

$$(3.28) \quad \mathbf{M}_c(\mathbf{x}) = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\zeta}} = \begin{pmatrix} x_{p_2} - x_{p_1} & x_{p_3} - x_{p_1} \\ y_{p_2} - y_{p_1} & y_{p_3} - y_{p_1} \end{pmatrix} = \sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \mathbf{x}_p$$

with $|\mathbf{M}_c| = 2|\omega_c|$. The discrete gradient of a discrete scalar quantity $\phi_h = \phi_{p_1} \psi_{p_1} + \phi_{p_2} \psi_{p_2} + \phi_{p_3} \psi_{p_3}$ is then given by

$$(3.29) \quad \begin{aligned} \tilde{\nabla}_c^p \phi_p &= \mathbf{M}_c^{-T} \frac{\partial \phi_h}{\partial \boldsymbol{\zeta}} = \mathbf{M}_c^{-T} \sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \phi_p \\ &= \frac{1}{2|\omega_c|} \begin{pmatrix} (y_{p_2} - y_{p_3})\phi_{p_1} + (y_{p_3} - y_{p_1})\phi_{p_2} + (y_{p_1} - y_{p_2})\phi_{p_3} \\ (x_{p_3} - x_{p_2})\phi_{p_1} + (x_{p_1} - x_{p_3})\phi_{p_2} + (x_{p_2} - x_{p_1})\phi_{p_3} \\ 0 \end{pmatrix} = \nabla_c^p \phi_p. \end{aligned}$$

In the following we will denote a generic node of element ω_c by p , as usual, and the corresponding two nodes on the opposite edge are denoted in counterclockwise order by p^- and p^+ , respectively. Furthermore, we will denote the edge that connects p with p^- by e_{pc}^- and the edge that links p with p^+ by e_{pc}^+ . We now compute the single contribution of element ω_c and node p to the curl on the dual mesh $|\omega_p| \nabla_p^c \times \nabla_c^p \phi_p = -\sum_c l_{pc} \mathbf{n}_{pc} \times \nabla_c^p \phi_p$, i.e., the contribution $-l_{pc} \mathbf{n}_{pc} \times \nabla_c^p \phi_p$, which after a lengthy but straightforward calculation reduces to

$$(3.30) \quad -l_{pc} \mathbf{n}_{pc} \times \nabla_c^p \phi_p = \frac{1}{2} (\phi_{p^+} - \phi_{p^-}) = \phi_{e_{pc}^+} - \phi_{e_{pc}^-},$$

i.e., the difference of values at the respective edge midpoints. Consider now two adjacent elements ω_c and ω_d which share a common edge e composed of nodes p and q . We assume that the elements are ordered counterclockwise, i.e., the previous element is ω_c and the subsequent element is ω_d . The value at the edge midpoint given by the previous element ω_c is denoted by $\phi_e^- = \phi_{e_{pc}^+}$ and the value at the edge midpoint given by the subsequent element ω_d is denoted by $\phi_e^+ = \phi_{e_{qd}^-}$. Since the values of the scalar field ϕ at the edge midpoint are the *same* for two adjacent elements that share a common edge e , i.e., $\phi_e^+ = \phi_e^- = \frac{1}{2}(\phi_p + \phi_q)$, summing up over all elements ω_c around the node p yields zero, due to the telescopic sum property:

$$(3.31) \quad |\omega_p| \nabla_p^c \times \nabla_c^p \phi_p = - \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \times \nabla_c^p \phi_p = \sum_{e \in \mathcal{E}(p)} \phi_e^+ - \phi_e^- = 0.$$

Here, $\mathcal{E}(p)$ is the set of edges connected to node p . The above identity completes the proof that the discrete curl on the dual mesh applied to the discrete gradient on the primal mesh is zero. $\square$

This lemma has already been proven previously in a different setting in [46] and [4].

THEOREM 3.2. *Let us assume periodic boundary conditions or the boundary conditions $\int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, ds = 0$, $\int_{\partial\Omega} \boldsymbol{\sigma} \cdot \mathbf{n} \, ds = 0$, and $\int_{\partial\Omega} \boldsymbol{\beta} \cdot \mathbf{n} \, ds = 0$. Then, the semidiscrete scheme (3.5) with node values given by (3.9), (3.10), and (3.11) and the correction factor given by (3.12) and (3.14) is nonlinearly stable in the sense that it conserves total energy:*

$$(3.32) \quad \int_{\Omega} \frac{dE}{dt} \, d\mathbf{x} = 0.$$

Proof. The semidiscrete total energy conservation law can be obtained by contracting the semidiscrete scheme (3.5) with the discrete main field $\mathbf{w}_c = (-p_c, \mathbf{v}_c, T_c, \boldsymbol{\Gamma}_c, \boldsymbol{\beta}_c)^T$. Summing up over all c and exchanging summation over p and c in the fluctuation terms we thus obtain

$$\begin{aligned} & \sum_c m_c \left(-p_c \frac{d\tau_c}{dt} + \mathbf{v}_c \frac{d\mathbf{v}_c}{dt} + T_c \frac{dS_c}{dt} + \boldsymbol{\Gamma}_c : \frac{d\mathbf{A}_c}{dt} + \boldsymbol{\beta}_c \cdot \frac{d\mathbf{J}_c}{dt} \right) \\ & + \sum_p \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (p_c(\mathbf{v}_p - \mathbf{v}_c) + \mathbf{v}_c(p_p - p_c) + (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_c)\mathbf{v}_c + T_c(\rho_p \boldsymbol{\beta}_p - \rho_c \boldsymbol{\beta}_c)) \\ & + \sum_p \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot (\rho_c \boldsymbol{\Gamma}_c \mathbf{A}_c(\mathbf{v}_p - \mathbf{v}_c) + \rho_c \boldsymbol{\beta}_c \otimes \mathbf{J}_c(\mathbf{v}_p - \mathbf{v}_c) + \rho_c \boldsymbol{\beta}_c(T_p - T_c)) \\ & + \sum_p \sum_{c \in \mathcal{C}(p)} l_{pc} (\alpha_p + \epsilon_p) (\mathbf{v}_c - \mathbf{v}_p) \cdot \mathbf{v}_c \end{aligned}$$

$$\begin{aligned}
& + \sum_p \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \cdot \rho_c \boldsymbol{\beta}_c (\alpha_p + \epsilon_p) |\omega_p| \nabla_p^c \cdot (\rho_c \boldsymbol{\beta}_c) \\
& = \sum_c \left(T_c \Pi_c + T_c \pi_c - \frac{1}{\theta_2(\tau_2)} \boldsymbol{\beta}_c \cdot \boldsymbol{\beta}_c - \frac{1}{\theta_1(\tau_1)} \boldsymbol{\Gamma}_c : \boldsymbol{\Gamma}_c \right).
\end{aligned}$$

The term in the first bracket is obviously the time derivative of the total energy in cell c . Now, let us consider the definition of α_p given in (3.14) and (3.12), and let us recall the hypothesis on the boundary conditions. Furthermore, notice that the sum over all points p of the boundary integrals on the dual volumes shown in (3.12) cancels out. Then, making use of the definition of the entropy production terms Π_c and π_c , the above expression reduces to

$$(3.33) \quad \sum_c m_c \frac{dE_c}{dt} = \int_{\Omega} \frac{dE}{dt} d\mathbf{x} = 0,$$

which completes the proof. $\square$

THEOREM 3.3. *The semidiscrete scheme (3.5) preserves the determinant constraint $|\mathbf{A}_c| = \rho_c(t)/\rho_c(0)$ exactly on the semidiscrete level in the sense*

$$(3.34) \quad \frac{d|\mathbf{A}_c|}{dt} = \frac{d}{dt} \left(\frac{\rho_c}{\rho_c(0)} \right) = -\frac{\rho_c^2}{\rho_c(0)} \frac{d\tau_c}{dt} = -\frac{|\mathbf{A}_c|}{|\omega_c|} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \cdot (\mathbf{v}_p - \mathbf{v}_c).$$

Proof. The source term in (3.5d) has been designed in [39] not to affect mass conservation, thus it vanishes when considering the time evolution equation of $|\mathbf{A}_c|$. Because of the pointwise discretization of the source term, the scheme (3.5d) yields by construction a zero source term in the time evolution of $|\mathbf{A}_c|$; see the proof in [12]. Therefore, the source term in (3.5d) can be neglected. Division of (3.5d) by $m_c = \rho_c |\omega_c|$ leads to

$$(3.35) \quad \frac{d\mathbf{A}_c}{dt} = -\frac{\mathbf{A}_c}{|\omega_c|} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} (\mathbf{v}_p - \mathbf{v}_c).$$

Using the Jacobi identity for the time derivative of the determinant of $\mathbf{A}_c$ and using the discretized PDE for $\tau_c = \rho_c^{-1}$ given by (3.5a) yields the sought result, which is the discrete counterpart of (2.4):

$$\begin{aligned}
(3.36) \quad \frac{d|\mathbf{A}_c|}{dt} &= |\mathbf{A}_c| \operatorname{tr} \left(\mathbf{A}_c^{-1} \frac{d\mathbf{A}_c}{dt} \right) \\
&= -|\mathbf{A}_c| \operatorname{tr} \left(\mathbf{A}_c^{-1} \mathbf{A}_c \frac{1}{|\omega_c|} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} (\mathbf{v}_p - \mathbf{v}_c) \right) \\
&= -\frac{|\mathbf{A}_c|}{|\omega_c|} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \cdot (\mathbf{v}_p - \mathbf{v}_c) = -\frac{\rho_c^2}{\rho_c(0)} \frac{d\tau_c}{dt} = \frac{d}{dt} \left(\frac{\rho_c}{\rho_c(0)} \right). \quad \square
\end{aligned}$$

THEOREM 3.4. *Let us assume a vanishing source, i.e., $\tau_1 \rightarrow \infty$, and initially curl-free data for the distortion field given by $\mathbf{A}_c(0) = \nabla_c^p \boldsymbol{\xi}_p$. Thus one has $\nabla_p^c \times \mathbf{A}_c(0) = 0$. Let us assume exact time integration. Then, the semidiscrete scheme (3.5) preserves the curl of the rows of $\mathbf{A}_c$ exactly at the semidiscrete level in the sense*

$$(3.37) \quad \nabla_p^c \times \mathbf{A}_c(t) = 0.$$

Proof. In the absence of source terms ($\tau_1 \rightarrow \infty$) the distortion field represents the inverse deformation gradient, i.e., the Eulerian gradient of the Lagrangian reference coordinates,

$$(3.38) \quad \mathbf{A}_c = \nabla_c^p \boldsymbol{\xi}_p = \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \boldsymbol{\xi}_p \right) \mathbf{M}_c^{-1},$$

which is in particular true for the initial time $t = 0$. Division of (3.5d) by $m_c = \rho_c |\omega_c|$ and using $\sum_p l_{pc} \mathbf{n}_{pc} \mathbf{v}_c = 0$ yields

$$(3.39) \quad \frac{d\mathbf{A}_c}{dt} + \mathbf{A}_c \frac{1}{\omega_c} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \mathbf{v}_p = 0$$

or, in more compact notation,

$$(3.40) \quad \frac{d\mathbf{A}_c}{dt} + \mathbf{A}_c \nabla_c^p \mathbf{v}_p = 0.$$

Applying the Lagrangian time derivative to the definition (3.38) yields

$$(3.41) \quad \frac{d\nabla_c^p \boldsymbol{\xi}_p}{dt} = \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \boldsymbol{\xi}_p \right) \frac{d\mathbf{M}_c^{-1}}{dt} = - \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \boldsymbol{\xi}_p \right) \mathbf{M}_c^{-1} \frac{d\mathbf{M}_c}{dt} \mathbf{M}_c^{-1},$$

where we have employed the formula for the time derivative of the inverse of a matrix and where we have also used the fact that the only object which depends on time is actually $\mathbf{M}_c$ due to the moving mesh. Indeed, at the aid of the discrete trajectory (3.7) and thanks to (3.28) we obtain the following time derivative of $\mathbf{M}_c$:

$$(3.42) \quad \frac{d\mathbf{M}_c}{dt} = \sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \frac{d\mathbf{x}_p}{dt} = \sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \mathbf{v}_p.$$

Inserting (3.42) into (3.41) and using (3.38) leads to

$$(3.43) \quad \frac{d\nabla_c^p \boldsymbol{\xi}_p}{dt} = - \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \boldsymbol{\xi}_p \mathbf{M}_c^{-1} \right) \left(\sum_{q \in \mathcal{P}(c)} \frac{\partial \psi_q}{\partial \boldsymbol{\zeta}} \mathbf{v}_q \mathbf{M}_c^{-1} \right) = - \nabla_c^p \boldsymbol{\xi}_p \nabla_c^q \mathbf{v}_q,$$

which corresponds to (3.40) and thus to the numerical scheme (3.5d) used for the discretization of $\mathbf{A}_c$. We thus conclude that (3.5d) is equivalent to

$$(3.44) \quad \frac{d\mathbf{A}_c}{dt} = \frac{d\nabla_c^p \boldsymbol{\xi}_p}{dt}.$$

This means that, assuming exact time integration, $\mathbf{A}_c = \nabla_c^p \boldsymbol{\xi}_p$ for all times, since this relation is assumed to be true at the initial time $\mathbf{A}_c(0) = \nabla_c^p \boldsymbol{\xi}_p(0)$. Application of the discrete curl operator on the dual mesh to the previous relation leads to

$$(3.45) \quad \nabla_p^c \times \mathbf{A}_c = \nabla_p^c \times \nabla_c^q \boldsymbol{\xi}_q = 0,$$

thanks to the discrete vector calculus identity (3.21), which completes the proof. $\square$

THEOREM 3.5. *Let us assume a vanishing source, i.e., $\tau_2 \rightarrow \infty$, and initially curl-free data for the thermal impulse given by the discrete gradient of a scalar potential Z as $\mathbf{J}_c(0) = \nabla_c^p Z_p(0)$. It thus satisfies $\nabla_p^c \times \mathbf{J}_c(0) = 0$. Let us assume exact time integration. Then, the semidiscrete scheme (3.5) preserves the curl-free condition of $\mathbf{J}_c$ exactly at the semidiscrete level for all times, i.e.,*

$$(3.46) \quad \nabla_p^c \times \mathbf{J}_c = 0.$$

Proof. Dividing (3.5e) by $m_c = \rho_c |\omega_c|$, introducing the auxiliary scalar field $\lambda_p = T_p + \alpha_p |\omega_p| \nabla_p^c \cdot (\rho_c \beta_c)$, and using the fact that $\sum_p l_{pc} \mathbf{n}_{pc} T_c = 0$ and $\sum_p l_{pc} \mathbf{n}_{pc} \mathbf{v}_c = 0$ one obtains

$$(3.47) \quad \frac{d\mathbf{J}_c}{dt} + \mathbf{J}_c \cdot \frac{1}{\omega_c} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \mathbf{v}_p + \frac{1}{\omega_c} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \lambda_p = 0$$

or, in more compact notation,

$$(3.48) \quad \frac{d\mathbf{J}_c}{dt} + \mathbf{J}_c \cdot \nabla_c^p \mathbf{v}_p + \nabla_c^p \lambda_p = 0.$$

We now define the discrete gradient of a discrete scalar potential Z ,

$$(3.49) \quad \mathbf{J}_c = \nabla_c^p Z_p = \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} Z_p(t) \right) \cdot \mathbf{M}_c^{-1},$$

which is in particular assumed to hold at the initial time $t = 0$. We furthermore define

$$(3.50) \quad \frac{dZ_p}{dt} = -\lambda_p.$$

In the relation (3.49), both Z_p and $\mathbf{M}_c$ are functions of time. Applying the Lagrangian time derivative yields

$$(3.51) \quad \begin{aligned} \frac{d\nabla_c^p Z_p}{dt} &= \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} Z_p(t) \right) \cdot \frac{d\mathbf{M}_c^{-1}}{dt} + \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \frac{dZ_p}{dt} \right) \cdot \mathbf{M}_c^{-1} \\ &= - \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} Z_p(t) \cdot \mathbf{M}_c^{-1} \right) \cdot \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \mathbf{v}_p \mathbf{M}_c^{-1} \right) \\ &\quad - \left(\sum_{p \in \mathcal{P}(c)} \frac{\partial \psi_p}{\partial \boldsymbol{\zeta}} \lambda_p \right) \cdot \mathbf{M}_c^{-1} = -\nabla_c^p Z_p \cdot \nabla_c^q \mathbf{v}_q - \nabla_c^p \lambda_p, \end{aligned}$$

which is identical to (3.48) and thus to (3.5e) since $\mathbf{J}_c(0) = \nabla_c^p Z_p(0)$ holds. As such we can conclude that $\mathbf{J}_c = \nabla_c^p Z_p$. Application of the discrete curl operator and assuming exact time integration leads to the sought result

$$(3.52) \quad \nabla_p^c \times \mathbf{J}_c = \nabla_p^c \times \nabla_c^q Z_q = 0. \quad \square$$

4. Numerical results. We present a set of test cases that show the accuracy of the new Lagrangian HTC scheme for the GPR model, which will be labeled as Lag-HTC. Inviscid and viscous fluids are considered as well as ideal elastic solids.

For comparison, the Lagrangian scheme introduced in [8] is used to run some of the test cases, which is an extension of the EUCCLHYD scheme [35] to the GPR model. The detailed EOSs and the functions $\theta_1(\tau_1)$ and $\theta_2(\tau_2)$ used in this paper are given in [21, 10]. The polytropic index of the gas is assumed to be $\gamma = 7/5$ and the specific heat at constant volume is $c_v = 2.5$. Whenever a finite viscosity coefficient μ is specified, the relaxation time τ_1 is computed according to $\mu = \frac{1}{6}\rho_0 c_s^2 \tau_1$. Likewise, if a finite heat conduction coefficient $\kappa \neq 0$ is prescribed, the corresponding relaxation time τ_2 is evaluated from the asymptotic relation $\kappa = \rho_0 T_0 c_h^2 \tau_2$. In the other cases, no source terms are considered, thus we set $\tau_1 = \tau_2 = 10^{20}$. The distortion matrix is always initialized as $\mathbf{A} = \mathbf{I}$ since $\mathbf{x}(0) = \boldsymbol{\xi}$, and the thermal impulse is initially $\mathbf{J} = \mathbf{0}$. The reference density is given by $\rho_c(0) = \rho(\mathbf{x}_c, t = 0)$. If not stated otherwise, we set CFL = 0.05 and a six-stage fifth order Runge–Kutta scheme is adopted for time integration, thus ensuring that the Geometric Conservation Law [36] and total energy conservation are satisfied up to sufficient accuracy at the fully discrete level. For a quantitative study of the influence of the time integration scheme on the discrete structural properties of the model, we refer to [12]. In the numerical simulations, we monitor the following properties over time.

- Curl operators: the error related to the curl of the thermal impulse vector $\mathbf{J}$ is computed at a given time $t = t^n$ by considering

$$\epsilon^{\nabla \times \mathbf{J}} := \max_p \left| -\frac{1}{|\omega_p|} \sum_{c \in \mathcal{C}(p)} l_{pc} \mathbf{n}_{pc} \times \mathbf{J}_c \right|.$$

The error related to the curl of the distortion matrix $\mathbf{A}$ is computed by applying the above relation to each row of $\mathbf{A}$, and then taking the maximum norm.

- Determinant of $\mathbf{A}$: the error related to the geometric constraint $|\mathbf{A}_c| = \rho_c(t)/\rho_c(0)$ is computed at each time step as

$$\epsilon^{\det(\mathbf{A})} := \max_c \left| -\frac{|\mathbf{A}_c|}{|\omega_c|} \sum_{p \in \mathcal{P}(c)} l_{pc} \mathbf{n}_{pc} \cdot (\mathbf{v}_p - \mathbf{v}_c) + \frac{\rho_c^2}{\rho_c(0)} \frac{d\tau_c}{dt} \right|.$$

- Total energy conservation: the error at each time step is given by

$$\epsilon^E := \left| \sum_c \mathbf{w}_c \cdot \frac{d\mathbf{q}}{dt} \right|.$$

4.1. Numerical convergence study. We solve the isentropic vortex problem [32] to assess the accuracy of the scheme. We set $c_s = 0$ and $c_h = 0$ to model the inviscid Euler equations of compressible gas dynamics. The computational domain is $\Omega(0) = [0; 10]^2$ with slip wall boundaries everywhere. The initial condition is detailed in [32], and here the vortex is not transported, thus the convective velocity is set to zero. This test case is run until the final time $t_f = 1$ on a sequence of successively refined meshes of characteristic mesh size $h = \max_{\Omega(t_f)} \frac{|\omega_c|}{|\partial \omega_c|}$. The experimental order of convergence (EOC) is reported in Table 1, showing that the expected first order of accuracy is retrieved. The errors are measured in the L_2 norm for density, horizontal velocity, and pressure. Figure 2 shows the entropy and the energy distribution at time $t_f = 1$ on a computational mesh composed of $N_c = 5180$ cells. The time evolution of the curl of $\mathbf{A}$ and $\mathbf{J}$ are depicted in Figure 3, where we also monitor the error related

TABLE 1

EOC for the isentropic vortex problem using the Lag-HTC scheme. The errors are measured in the L_2 norm and refer to the variables ρ (density), u (horizontal velocity), and p (pressure) at time $t_f = 1$.

h	$L_2(\rho)$	EOC(ρ)	$L_2(u)$	EOC(u)	$L_2(p)$	EOC(p)
3.254E-01	2.6883E-01	-	1.4736E-01	-	3.6822E-01	-
2.490E-01	2.1588E-01	0.82	1.1103E-01	1.06	2.9549E-01	0.82
1.654E-01	1.4773E-01	0.93	7.3479E-02	1.01	2.0149E-01	0.94
1.283E-01	1.1088E-01	1.13	5.6521E-02	1.03	1.5117E-01	1.13

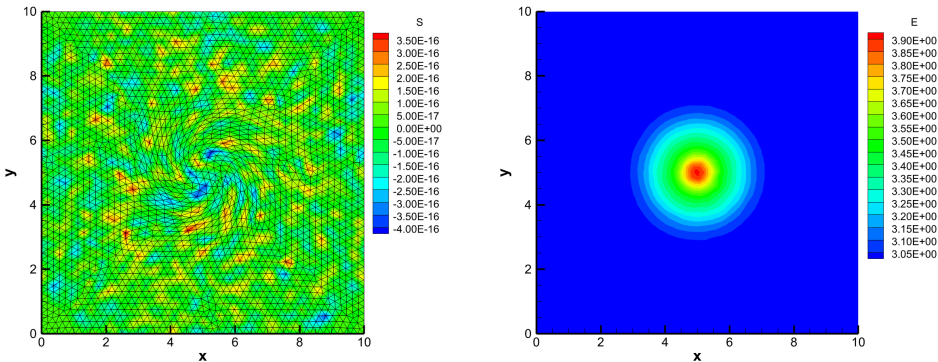


FIG. 2. Isentropic vortex problem at time $t_f = 1$. Left: entropy distribution and mesh configuration. Right: energy distribution. Note: color appears only in the online article.

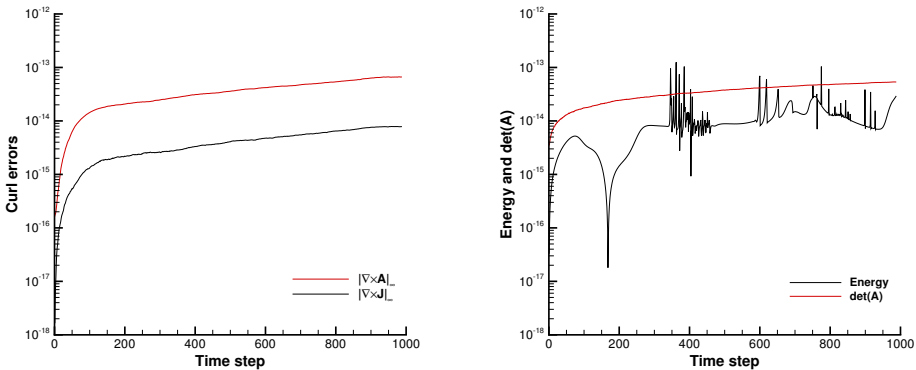


FIG. 3. Isentropic vortex problem. Left: time evolution of the curl of $\mathbf{A}$ and $\mathbf{J}$. Right: monitoring of the determinant of $\mathbf{A}$ compatibility and total energy conservation error. Note: color appears only in the online article.

to the determinant of $\mathbf{A}$ and the total energy conservation. The curl-free and the algebraic constraints are well preserved by the scheme. As expected, the flow remains exactly isentropic, while conserving at the same time total energy.

4.2. Viscous shock wave. We model compressible heat-conducting viscous flows by setting $c_s = c_h = 20$, $\mu = 2 \cdot 10^{-2}$, and $\kappa = 9.3333 \cdot 10^{-2}$. The initial computational domain is $\Omega(0) = [0; 1] \times [0; 0.2]$, discretized with triangles of size $h = 1/100$. No-slip wall boundaries are imposed in the y -direction, while Dirichlet

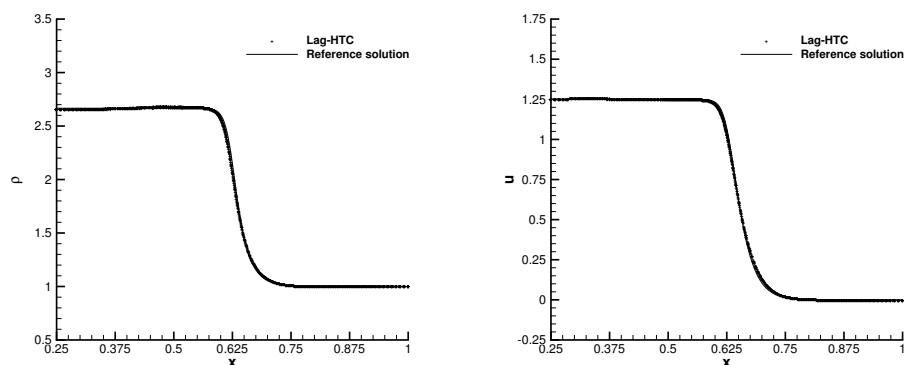


FIG. 4. Viscous shock problem at time $t_f = 0.2$. Scatter plot for density (left) and horizontal velocity (right) compared against the exact solution.

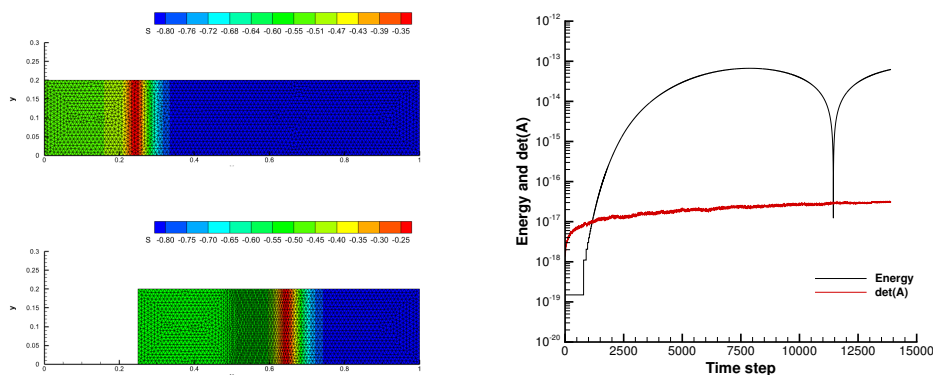


FIG. 5. Viscous shock problem. Left: mesh configuration and entropy distribution at the initial time (top) and at the final time (bottom). Right: monitoring of the determinant of $\mathbf{A}$ compatibility and total energy conservation error. Note: color appears only in the online article.

boundaries are prescribed in the x -direction. For Prandtl number $\text{Pr} = 0.75$, there exists an exact solution; see [7]. The Reynolds number is $\text{Re}_s = \rho_0 c_0 M_s L \mu^{-1}$, with the reference length $L = 1$. The initial condition is a shock centered at $x = 0.25$, propagating at Mach $M_s = 2$ with $\text{Re}_s = 100$. The upstream shock state is defined by $(\rho, u, v, p) = (\rho_0, 0, 0, 1/\gamma)$, with $c_0 = 1$ and $\rho_0 = 1$. The comparison between the numerical solution obtained with the Lagrangian HTC scheme and the exact solution of the compressible Navier–Stokes equations at time $t_f = 0.2$ is shown in Figure 4. A very good agreement is achieved for all quantities. The initial and final mesh configuration with the entropy distribution are depicted in Figure 5, where we see that entropy is increasing while the shock is moving into the fluid. Furthermore, we also plot the error related to the determinant of $\mathbf{A}$ and the total energy conservation. Both constraints are preserved up to machine precision.

4.3. Solid rotor problem. The solid rotor problem [10, 15] deals with ideal elastic solids. We fix $c_s = c_h = 1$ and the final time of the simulation is $t_f = 0.3$. The initial computational domain is $\Omega(0) = [-1; 1]^2$ with slip wall boundaries, which is discretized with a characteristic mesh size of $h = 1/150$ yielding a total number of $N_c = 94754$ cells. The initial condition of the material reads

$$(4.1) \quad (\rho, u, v, p) = \begin{cases} (1, -y/R, x/R, 1), & r < R, \\ (1, 0, 0, 1), & r \geq R, \end{cases} \quad t = 0, \quad \mathbf{x} \in \Omega(0),$$

with the initial discontinuity located at $R = 0.2$ and $r = \sqrt{x^2 + y^2}$. We run three different simulations using the new Lag-HTC scheme, and the first and second order EUCCLHYD scheme for the GPR model [8]. Figure 6 plots the distribution of the horizontal velocity component and the pressure at the final time. We can notice that the less diffusive result is achieved by the Lag-HTC scheme which is comparable (if not even less dissipative) to the second order EUCCLHYD scheme. The first order EUCCLHYD scheme leads to a much smoother solution, hence losing precision. Figure 7 shows the time evolution of the curl of $\mathbf{A}$ and $\mathbf{J}$ as well as the error related to the determinant of $\mathbf{A}$ and the total energy conservation over time. We can conclude that all the properties of the semidiscrete Lag-HTC scheme are fulfilled also for this test case.

4.4. Elastic vibration of a beryllium plate. This test case describes the elastic vibration of a beryllium plate; see [33, 45]. The domain is $\Omega(0) = [-0.03; 0.03] \times [-0.005; 0.005]$ and the length of the plate is $L = 0.06$. The mesh is composed of $N_c = 8414$ elements, and free-traction boundary conditions are set everywhere. The final time is chosen as $t_f = 3 \cdot 10^{-5}$, so that approximately one oscillating period is completed. We use a stiffened gas equation of state with $\gamma = 1.11$ and $\Gamma = 1.124$. The other parameters of the GPR model are: $c_s = 9046.59$, $c_0 = 12870$, $c_h = 0$, $c_v = 1$. The material initially has uniform density $\rho(0) = 1$ and pressure $p(0) = 0$, and is perturbed via a vertical velocity field of the form

$$(4.2) \quad v(x) = A\omega [a_1(\sinh(x') + \sin(x')) - a_2(\cosh(x') + \cos(x'))],$$

where $x' = \alpha(x + L/2)$, $\alpha = 78.834$, $A = 4.3369 \times 10^{-5}$, $\omega = 2.3597 \times 10^5$, $a_1 = 56.6368$, and $a_2 = 57.6455$. We show the entropy and pressure distribution in Figure 8 at different times. One can notice that the purely elastic behavior of the solid does not produce any entropy within our new Lag-HTC scheme. Figure 9 confirms the preservation of the curl-free property as well as the geometric relation on $\det(\mathbf{A})$ and total energy conservation over more than 170,000 time steps. Finally, this test problem is run with the first and second order EUCCLHYD scheme devised in [8]. In Figure 10 we show the time evolution of the vertical displacement of the point originally located at $\mathbf{x} = (0, 0)$. The first and second order EUCCLHYD schemes are compared against the new first order Lag-HTC scheme, showing that the thermodynamically compatible method substantially reduces numerical dissipation and the results are comparable to a second order EUCCLHYD scheme.

5. Conclusions. In this paper we have presented a new numerical method for the solution of the unified GPR model of continuum mechanics. The scheme preserves all essential structural properties of the governing PDE system exactly also on the discrete level. More precisely, the method is by construction compatible with the second principle of thermodynamics, since the entropy inequality is directly discretized within our scheme. Furthermore, thanks to a judicious choice of the nodal fluxes, the method also conserves total energy at the semidiscrete level and thus is also compatible with the first principle of thermodynamics. The fact that we directly discretize the entropy inequality and obtain the energy conservation law as a consequence is in line with the SHTC formalism of Godunov and Romenski, but is still rather unconventional at the discrete level. In addition, our new schemes are also by construction exactly compatible with the determinant constraint on the distortion

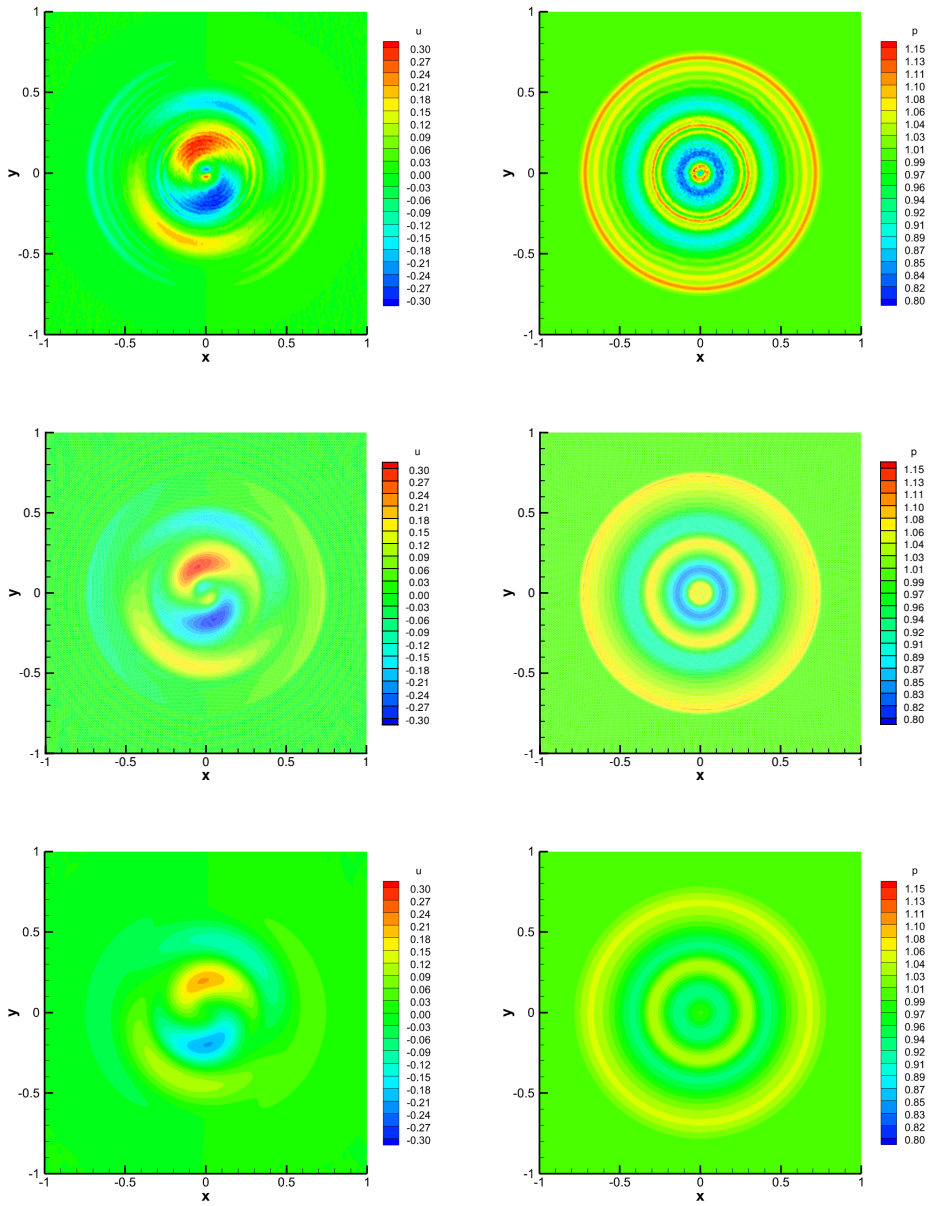


FIG. 6. Solid rotor problem at time $t_f = 0.3$. Horizontal velocity (left column) and pressure (right column) distribution. Top: Lag-HTC scheme. Middle: second order EUCCLHYD scheme [8]. Bottom: first order EUCCLHYD scheme [8]. 21 contours are used for the horizontal velocity (left column) in the interval $[-0.3; 0.3]$ and for the pressure (right column) in the interval $[0.8; 1.15]$. Note: color appears only in the online article.

field. This algebraic geometric identity is not easy to achieve at the discrete level. In our scheme, the discrete compatibility is possible thanks to a judicious choice of nodal fluxes and compatible nabla operators. Last but not least, in the absence of algebraic source terms, the method presented in this paper also preserves the curl-free property

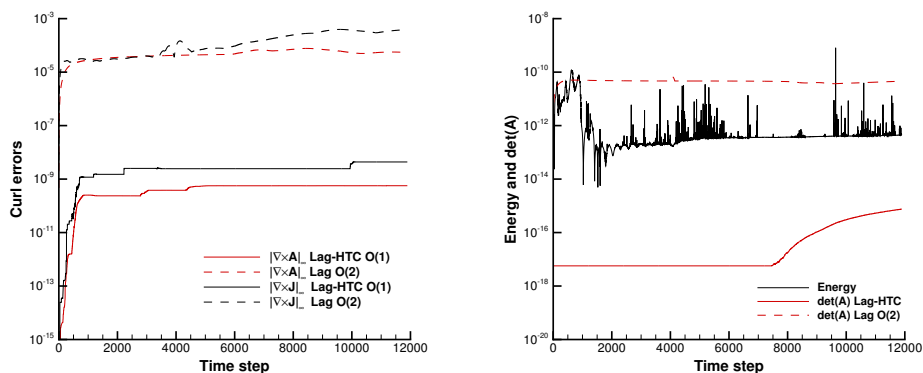


FIG. 7. Solid rotor problem. Left: time evolution of the curl of $\mathbf{A}$ and $\mathbf{J}$. Right: monitoring of the determinant of $\mathbf{A}$ compatibility and total energy conservation error. Comparison with the second order EUCCLHYD scheme of [8]. Note: color appears only in the online article.

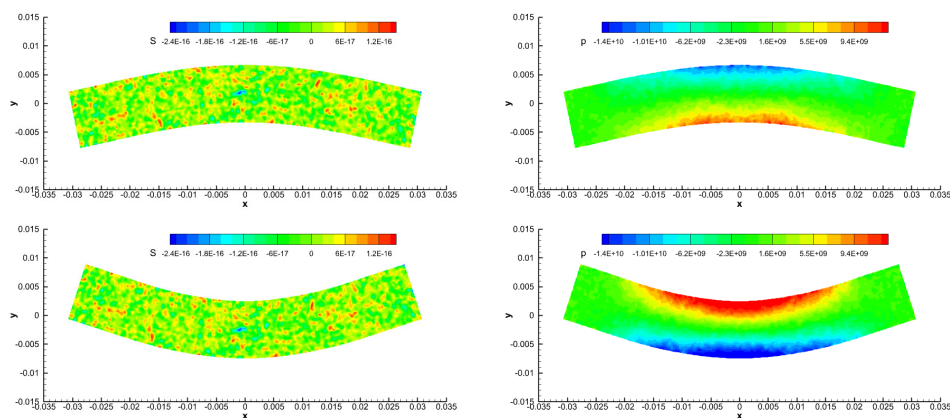


FIG. 8. Beryllium plate problem. Entropy (left) and pressure (right) distribution at time $t = 1 \cdot 10^{-5}$ (top) and $t = 2 \cdot 10^{-5}$ (bottom). Note: color appears only in the online article.

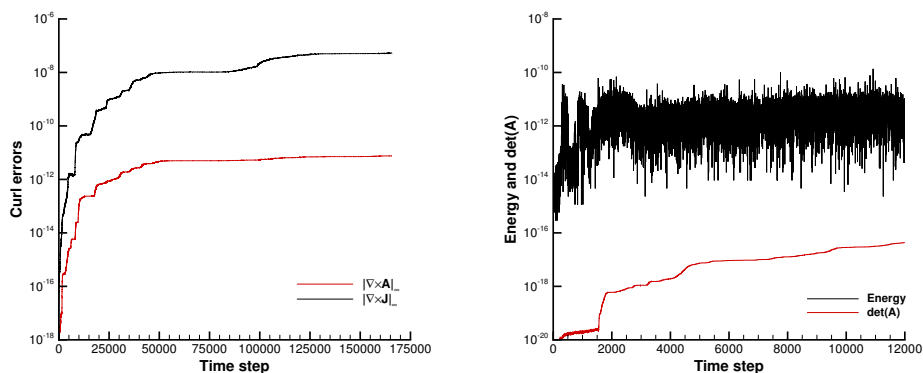


FIG. 9. Beryllium plate problem. Left: time evolution of the curl of $\mathbf{A}$ and $\mathbf{J}$. Right: monitoring of the determinant of $\mathbf{A}$ compatibility and total energy conservation error. Note: color appears only in the online article.

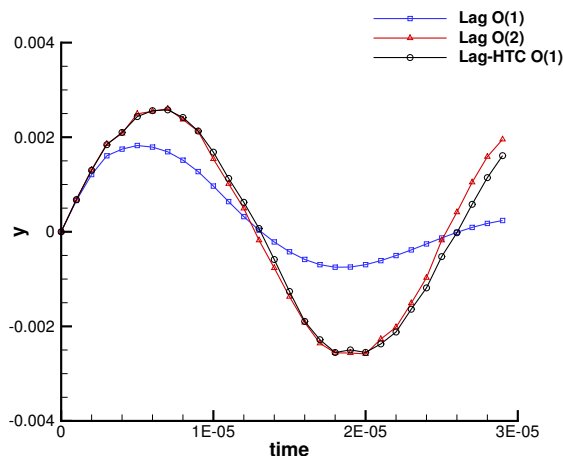


FIG. 10. *Beryllium plate problem.* Vertical displacement of the point initially located at $\mathbf{x} = (0, 0)$ with first and second order *EUCCLHYD* scheme [8] as well as with the new *Lag-HTC* scheme. Note: color appears only in the online article.

of the distortion field $\mathbf{A}$ and of the thermal impulse $\mathbf{J}$ exactly at the semidiscrete level. As such, all basic structural properties of the PDE system are satisfied by our discretization. The numerical results obtained for some basic test cases underline that the properties are also achieved in practice. Future work will concern the extension to higher order of accuracy, following the approach recently outlined in [4].

Appendix A. Nondimensional form of the GPR model. The physical quantities involved in the GPR model (2.1) have the following SI units:

$$[\rho] = \frac{\text{kg}}{\text{m}^3}, \quad [\mathbf{v}] = \frac{\text{m}}{\text{s}}, \quad [S] = \frac{\text{m}^2}{\text{s}^2 \text{K}}, \quad [\mathbf{A}] = 1, \quad [\mathbf{J}] = \frac{\text{Ks}}{\text{m}},$$

$$[T] = \text{K}, \quad [p] = \frac{\text{kg}}{\text{s}^2 \text{m}}, \quad [E] = \frac{\text{m}^2}{\text{s}^2}, \quad [\tau_i] = \text{s}, \quad i \in \{1, 2\}, \quad [\mathbf{\Gamma}] = \frac{\text{m}^2}{\text{s}^2}, \quad [\beta] = \frac{\text{m}^3}{\text{s}^3 \text{K}}$$

with $[\mathbf{x}] = \text{m}$ and $[t] = \text{s}$. Furthermore, the quantities related to the shear sound speed and the finite propagation speed of heat waves have the following units:

$$(A.1) \quad [c_s] = \frac{\text{m}}{\text{s}}, \quad [c_h] = \frac{\text{m}^2}{\text{s}^2 \text{K}}.$$

System (2.1) can be rescaled relying on the following nondimensional variables:

$$(A.2) \quad \tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{\mathbf{v}} = \frac{\mathbf{v}}{v_0}, \quad \tilde{S} = \frac{\rho_0}{p_0 T_0} S, \quad \tilde{\mathbf{J}} = \frac{x_0}{T_0 t_0} \mathbf{J},$$

$$\tilde{T} = \frac{T}{T_0}, \quad \tilde{p} = \frac{p}{p_0}, \quad \tilde{E} = \frac{\rho_0}{p_0} E, \quad (\tilde{\tau}_1, \tilde{\tau}_2) = \frac{(\tau_1, \tau_2)}{t_0}, \quad \tilde{x} = \frac{x}{x_0}, \quad \tilde{t} = \frac{t}{t_0},$$

where $(x_0, t_0, \rho_0, v_0, T_0, p_0)$ are typical values referred to the problems under consideration. Notice also that $v_0 = x_0/t_0$, $T_0 = p_0/\rho_0$ and $p_0/\rho_0 = v_0^2$. The nondimensional form of the GPR model is then given by

$$(A.3a) \quad \frac{1}{t_0} \tilde{\rho} \frac{d\tilde{\tau}}{d\tilde{t}} - \frac{v_0}{x_0} \frac{\partial \tilde{v}_k}{\partial \tilde{x}_k} = 0,$$

$$(A.3b) \quad \frac{\rho_0 v_0}{t_0} \tilde{\rho} \frac{d\tilde{v}_i}{d\tilde{t}} + \frac{p_0}{x_0} \frac{\partial}{\partial \tilde{x}_k} \left(\tilde{p} \delta_{ik} + p_0 \tilde{\rho} A_{j,i} \tilde{\Gamma}_{j,k} + p_0 \tilde{\rho} \tilde{J}_i \tilde{\beta}_k \right) = 0,$$

$$(A.3c) \quad \frac{p_0}{T_0 t_0} \tilde{\rho} \frac{d\tilde{S}}{d\tilde{t}} + \frac{p_0}{T_0 t_0} \frac{\partial(\tilde{\rho}\tilde{\beta}_k)}{\partial\tilde{x}_k} = \frac{1}{T_0} \left(\frac{p_0^2}{\rho_0^2} \frac{\rho_0}{t_0 v_0^2} \frac{\tilde{\Gamma}_{ij}\tilde{\Gamma}_{ij}}{\tilde{T}\tilde{\theta}_1} \left(\frac{p_0 x_0}{\rho_0 T_0 t_0} \right)^2 \frac{\rho_0 T_0^2}{t_0 v_0^4} \frac{\tilde{\beta}_i \tilde{\beta}_i}{\tilde{T}\tilde{\theta}_2} \right) \geq 0,$$

$$(A.3d) \quad \frac{\rho_0}{t_0} \tilde{\rho} \frac{dA_{ik}}{d\tilde{t}} + \frac{\rho_0 v_0}{x_0} \tilde{\rho} A_{im} \frac{\partial \tilde{v}_m}{\partial \tilde{x}_k} = -\frac{p_0}{t_0 v_0^2} \frac{1}{\tilde{\theta}_1} \tilde{\Gamma}_{ik},$$

$$(A.3e) \quad \frac{\rho_0 T_0}{x_0} \tilde{\rho} \frac{d\tilde{J}_i}{d\tilde{t}} + \frac{\rho_0 T_0 t_0}{x_0} \frac{v_0}{x_0} \tilde{\rho} \tilde{J}_k \frac{\partial \tilde{v}_k}{\partial \tilde{x}_i} + \frac{\rho_0 T_0}{x_0} \tilde{\rho} \frac{\partial \tilde{T}}{\partial \tilde{x}_i} = -\frac{p_0 x_0}{\rho_0 T_0 t_0} \frac{\rho_0 T_0^2}{t_0 v_0^4} \frac{1}{\tilde{\theta}_2} \tilde{\beta}_i.$$

Introducing the parameter $\varepsilon = v_0^2/c_0^2$, with the reference speed of sound $c_0 = \sqrt{p_0/\rho_0}$, and omitting the tildes for readability, the nondimensional system finally reads

$$(A.4a) \quad \rho \frac{d\tau}{dt} - \frac{\partial v_k}{\partial x_k} = 0,$$

$$(A.4b) \quad \rho \frac{dv_i}{dt} + \frac{1}{\varepsilon} \frac{\partial}{\partial x_k} (p\delta_{ik} + \sigma_{ik}) = 0,$$

$$(A.4c) \quad \rho \frac{dS}{dt} + \frac{\partial(\rho\beta_k)}{\partial x_k} = \frac{\Gamma_{ij}\Gamma_{ij}}{T\theta_1(\tau_1)} + \frac{\beta_i\beta_i}{T\theta_2(\tau_2)} \geq 0,$$

$$(A.4d) \quad \rho \frac{dA_{ik}}{dt} + \rho A_{im} \frac{\partial v_m}{\partial x_k} = -\frac{1}{\theta_1(\tau_1)} \Gamma_{ik},$$

$$(A.4e) \quad \rho \frac{dJ_i}{dt} + \rho J_m \frac{\partial v_m}{\partial x_i} + \rho \frac{\partial T}{\partial x_i} = -\frac{1}{\theta_2(\tau_2)} \beta_i.$$

For $\varepsilon = 1$, the above system looks identical to the dimensional model (2.1). For compressible flows, the parameter $\varepsilon = M^2$ is related to the square of the Mach number, and we refer to [10] for further details.

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